Preliminary Report

Comments Welcomed

SIMULATION OF UNION HEALTH AND WELFARE FUNDS[†] - PART I

by

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PART I

1. Introduction

The trustees of multi-employer, jointly trusteed health and welfare funds in the building trades face particularly difficult financial problems because of the characteristics of the industry. The movement of both employer and employee from job to job and area to area, and the casual and variable nature of employment, mean that a man's eligibility for fund benefits cannot be based solely on a particular job. As a result, a system of rules has been established under which a man becomes eligible for benefits if he works a minimum number of hours during a given period. He may, by working the minimum number in a current period, become eligible for benefits at a future date, or he may have earned current benefits because of hours he has worked in the past. The minimum number of hours, the period during which they must be worked, and the length of time a worker remains eligible, vary greatly from fund to fund.

Essentially, the goals of fund trustees are to provide benefits for the "workers active at the trade," to use the maximum sums available for current benefits, and at the same time to maintain fund solvency. Their decisions concerning eligibility rules, the amount of money that will be spent for benefits per eligible worker, and the sum of money to be held in reserve determine the fund's financial position. Because of the fluctuations in employment in the building trades, however, the trustees can never predict in advance just how many men will be working, the number of hours they will work, and how many of them will be achieving eligibility. Consequently, they can never be sure exactly what their income or expenses will be at any one time.

The fund's financial problems affect the degree to which the trustees' will achieve their goals. When increased expense, reduced income, or inadequate reserves threaten the fund with insolvency, the trustees can retrieve the fund's position by changing the eligibility rules to reduce the number of beneficiaries, by shortening the period during which members remain eligible, or by cutting down on the benefits available. All of these solutions have been used by various funds, but each one reduces the effectiveness of the health and welfare plan. Changes in eligibility rules and reduction in benefits undermine the employees' confidence in the fund, and may lead to their purchase of coverage outside the fund. This in turn may cause unrest in the union and internal political difficulties. Even when a plan is running successfully, many members prefer to receive the amounts contributed by employers in cash rather than benefits. When their confidence in the benefit plan is undermined by a series of changes in eligibility and benefits, they will certainly prefer the cash. Frequent changes may, in fact, lead to the termination of the plan.

In rather simplified terms, the essence of fund management lies in control of the money flowing into and out of the fund. The flow of income into the fund during a particular period is determined largely by the contribution rate and the working hours to which the rate applies. It is also, of course, affected by earnings from investment of reserves and by reciprocal agreements between funds.

Money flows out of the fund primarily for the purchase or provision of benefits and for operating expenses. The amounts spent on benefits, assuming that these are purchased rather than provided directly, depend on the premium rate and the number of eligible workers. This last, in turn, depends upon the number of members who work the required number of hours called for by the eligibility rules. The operating expenses of the fund are determined by the size of the fund, the type of administration, the efficiency of the administrator, and the range of functions performed.

The timing of flows of money into and out of the fund is an important determinant of the fund's financial position. Since the eligibility rules generally provide for future benefit coverage on the basis of hours worked in a current period, theoretically current income pays for future benefits. The timing of money outflows is affected not only by the number of workers who become eligible, but by the length of time over which coverage is continued. The eligibility rule and the period over which benefits are payable create a future liability for the trustees, even though their legal liability continues only so long as there is money in the fund.

Not infrequently, more money is flowing out of the fund than is being replaced by income into it. For this reason a reserve sum of money is needed to serve as a reservoir which can be tapped to meet expenses, and to maintain solvency. The reservoir may also enable the trustees to adjust eligibility rules during periods of depressed employment conditions to provide for the continued coverage of workers active in the trade but not currently employed. In some funds this reservoir is created by provisions specifying that benefits will begin only after the fund has been collecting income for some period of time.

Inevitably, fund trustees find themselves in a sort of three-cornered tug-of-war as they seek a mixture of eligibility rules and benefit package which will balance their conflicting goals:

1. To provide the maximum amount of benefits possible per eligible employee;

2. To cover as many active workers in the trade as possible;

3. To ensure that the fund remains solvent.

The sources of conflict among the goals are obvious. If the eligibility rule is very stringent, few employees will be covered but the benefits per covered employee can be generous. Alternatively, if the eligibility rule is not made more restrictive, and yet the benefits offered eligible employees are expanded, the fund runs a higher risk of insolvency.

If the trustees are to trade off among these three goals most effectively, they must have a clear understanding of the way the elements involved interact. If, for example, workers are required to work ten more hours to become eligible for benefits, exactly how many employees would then fail to achieve eligibility? And how much would the odds of the fund's becoming insolvent be decreased? In sum, what the trustees need is a clear idea of the <u>quantitative</u> impact of a variety of combinations of eligibility rules and benefit costs. They need to know not only whether or not a change in expenditures for benefits or in the eligibility rule will achieve one of their goals to a greater or lesser degree, but they need also to know how much of each of their goals would be achieved or sacrificed were they to make the change. As matters now stand, however, this sort of careful analysis of the interactions is not ordinarily undertaken, and as a result many health and welfare plans are not meeting as well as they could and should the goals for which they were established.

Even though the trustees can make a multiplicity of choices among combinations of eligibility rules, expenditures for benefits, and fund reserve requirements, some aspects of the way these interact are evident. Increases in benefit expense, for example, will always mean increased expenditures for the fund if all other aspects of fund operation remain the same. And, given the usual spread of working hours among members, a reduction in the number of hours needed for eligibility will also mean increased expenditures because there will be a larger number of eligible workers. Because of the great number of possible combinations, and the wide variety of possible results, it is almost impossible for the trustees to know exactly how one change, or several changes, will affect all the other aspects of fund operation.

The simulation model described in this paper attempts to depict the essentials of fund behavior in such a fashion that the interactions described above may be determined. In order to carry out the analysis the following information is required:

1. The number of men active at the trade in the jurisdiction;

2. A history over time of the number of hours worked by the members;

3. The eligibility rule;

4. The contribution rate;

5. The premium rate;

6. Administrative expenses;

7. The amount of the fund reserve at the time of analysis.

1.1 Scope of Analysis

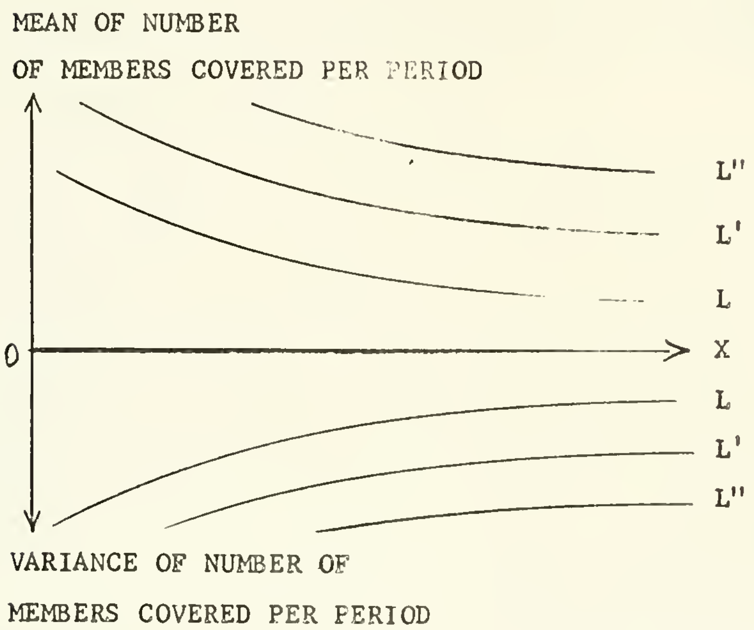
Of the many parameters which determine the behavior of the economic reserves in the fund, the trustees in fact control only the parameter (X,L) of the eligibility rule, the initial reserves U_o in the fund before benefit payments begin, the insurance premium r per member covered, and to some extent the employer contribution rate k. Our objective, which we will restate in slightly different language in 1.4, is to determine the way the probability of insolvency, and the mean and variance of members covered per time period vary with changes in these <u>controllable</u> parameters.

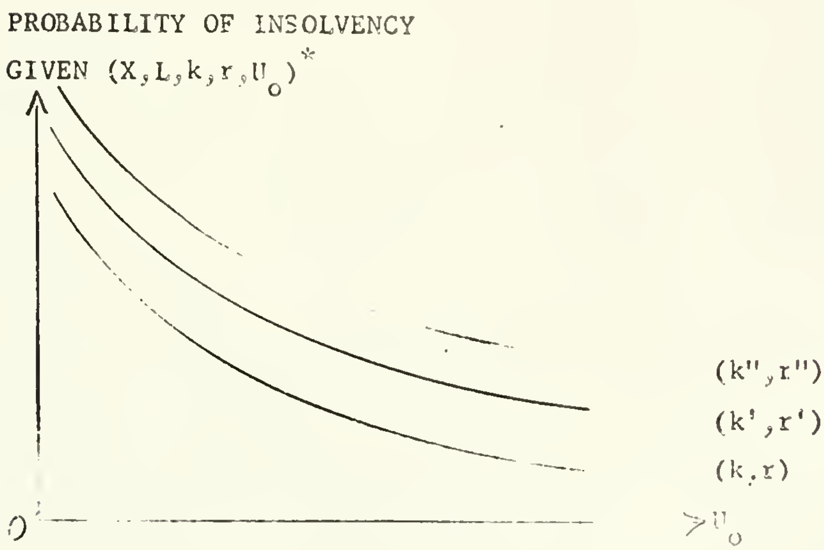
In Part I of this paper we describe a model of the random process generating hours worked per time period by each member active at the trade, and then show how the behavior of eligibility rules may be investigated using simulation techniques. The class of eligibility rules we examine in some detail may be described as follows:

> A member must work at least X hours during time periods t-L through t-1 in order to be covered during period t; t=1,2,... and L ≥ 1.

In Part II we present a series of graphs that describe how the probability of insolvency, the mean number of members covered per period and the variance of this mean vary with changes in values of controllable parameters. These graphs are derived by a mixture of mathematical analysis and monte carlo simulation, as the reader will subsequently see. Since (X,L) is the only controllable parameter influencing the mean and variance of members covered per

period, we display their behavior as a function of (X,L) independent of graphs of the probability of insolvency as a function of controllable parameters:





*One such graph is displayed for each (X,L) pair.

The mathematical analysis underlying these graphs is presented in Part 1. We define symbols in 1.2, set down the assumptions underlying the model generating "hours worked by the ith worker in period t" in 1.3, and restate objectives in more formal language in 1.4. Section 1.5 defines auxiliary symbols used in the following sections.

In section 2.1 we derive the mean vector and covariance matrix of a random vector $\underline{\tilde{\delta}}_t$, and use them to determine the mean and variance in any given period of the amount of economic reserves in the fund. The core of the simulation routine consists of generating values of $\underline{\tilde{\delta}}_t$ and then using these values to determine how the probability of insolvency varies with changes in parameter values. We also present the mean and variance of the average number of members covered per time period as a function of quantities calculated in 2.4 and 2.5. A mutivariate central limit theorem is applied to $\underline{\tilde{\delta}}_t$ in 2.2, demonstrating that $\underline{\tilde{\delta}}_t$ is asymptotically Multinormal; we exploit this fact later, and by doing so, greatly reduce computation time. Sections 2.3, 2.4, and 2.5 show how to calculate elements of the mean vector and covariance matrix of $\underline{\tilde{\delta}}_t$.

In section 3 we describe the structure of the simulation routine. Section 3.1 shows how the periodicity of elements of the covariance matrix of $\underline{\tilde{\delta}}_t$ may be exploited to enable us to find the square root $\underline{\underline{S}}$ of this matrix, no matter how large its order may be. (The matrix $\underline{\underline{S}}$ is used to transform sequence of standardized normal random deviates into values of $\underline{\tilde{\delta}}_t$). Finally, in section 3.2 we

outline the computer program used in the simulation.

Part II applies computer simulation routine and mathematical results of Part I to data from the a building trades union. Results are presented in the form of the graphs described earlier.

1.2 Definitions

Define

h_{it} - number of hours worked by the ith worker in period t, $i=1,2,\ldots,N$; $t=1,2,\ldots$,

I_t - income in the tth period,

α - expense in the tth period,

k - employer contribution rate per period,

n_t - number of employees covered in the tth period,

N - number of members active at the trade,

U_t - amount of reserves in the fund at the end of period t,

r - insurance premium per covered employee,

L - a parameter of the eligibility rule; the number of past periods over which hours are accrued in determining a worker's eligibility for any given period,

X - number of hours a worker must work during periods t-L through t-1 to be covered in the tth period.

Then

$$U_t = (U_0 - t\alpha) + \sum_{\tau=1}^{t} (I_\tau - rn_\tau) \quad , \tag{1a}$$

where

$$I_t = \sum_{i=1}^{N} k\, h_{it} \quad . \tag{1b}$$

Also define

$$\Omega = (\alpha, k, r, U_0, N, X, L) \quad ,$$

$P_\Omega(t) =$ $1 - P(\tilde{U}_\tau > 0 \mid$ [illegible] $)$ - the probability of insolvency within the time interval 0 to t given Ω;

$E_\Omega(t) =$ the expected waiting time to insolvency given Ω, conditional on insolvency occurring within the time interval 0 to t.

1.3 Assumptions

I The eligibility rule takes the form: a worker must work at least X hours during periods t-L through t-1 to be covered in period t, $t=1,2,\ldots$ and $L \geq 1$.

II The behavior of $\tilde{h}_{it}$ as t increases may be represented as

$$\tilde{h}_{it} = f(t) + \tilde{\epsilon}_{it} \quad , \quad \text{all } i \quad , \quad t=1,2,\ldots$$

where the random variables $\{\tilde{\epsilon}_{it}, \ i=1,2,\ldots,N; \ t=1,2,\ldots\}$ are mutually independent and identically distributed with mean 0 and variance σ_{ϵ}^2, and f(t) is some function of time.

III The number N of members active at the trade is constant over time periods $t=1,2,\ldots$.

1.4 Objectives

Given the ordered 7-tuple $(\alpha, k, r, U_o, N, X, L) \equiv \Omega$ and Assumptions I, II, and III what is:

(a) the expected waiting time $E_{\Omega}(t)$;

(b) the probability $P_{\Omega}(t)$;

(c) the marginal probability distribution of $\tilde{U}_t | \Omega$ at t;

(d) how do (a), (b), and (c) vary with changes in the elements of Ω?

(e) how do the mean and variance of $\tilde{n}_t$ vary with changes in the elements of Ω?

1.5 Further Definitions

It will be useful to re-express (1a) in the form

$$\tilde{U}_t = U_0 + \sum_{\tau=1}^{t} \tilde{\delta}_t \ , \tag{2}$$

where

$$\tilde{\delta}_t \equiv \tilde{U}_t - \tilde{U}_{t-i} \ , \tag{3}$$

Define for $t=1,2,\ldots$ and for $\tau=1,2,\ldots$

$$E(\tilde{\delta}_t) = \bar{\delta}_t \ ; \quad V(\tilde{\delta}_t) = \sigma^2_{\delta_t} \ , \quad \mathrm{Cov}(\tilde{\delta}_t,\tilde{\delta}_\tau) = \sigma_{t\tau}, \ t \neq \tau,$$

$$X_t = X - f(t) \ , \quad p_t = P\left(\sum_{\tau=t-L}^{t-1} \tilde{\epsilon}_{i\tau} > X - f(t) \right) = P\left(\sum_{\tau=t-L}^{t-1} \tilde{\epsilon}_{i\tau} > X_t \right) \ , \quad \text{all } i,$$

$$\zeta_t = -\alpha + f(t) - f(t-1) \ ,$$

$$\underline{\delta}_t = (\delta_1 \ \delta_2 \ \ldots \ \delta_\tau \ \ldots \ \delta_t)^t \ .^{\dagger}$$

2. Initial Results

2.1 Mean Vector and Variance-Covariance Matrix of $\tilde{\underline{\delta}}_t$

We will show that the random vector $\tilde{\underline{\delta}}_t$ is distributed with mean

$$\bar{\underline{\delta}}_t = (\bar{\delta}_1 \ \bar{\delta}_2 \ \ldots \ \bar{\delta}_\tau)^t$$

where

$$\bar{\delta}_\tau = \zeta_\tau - rNp_\tau \ , \quad \tau=.,2,\ldots,t \ , \tag{4}$$

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†We will indulge in abuse of notation and give the symbol t dual meaning. When it appears in a subscript it denotes time period t. As a superscript it denotes the transpose of a matrix or vector. The context will make the meaning clear.

and with symmetric covariance matrix

$$\underline{\underline{\Sigma}} = \begin{bmatrix} \sigma^2_{\delta_1} \cdots & \sigma_{1\tau} \cdots & \sigma_{1t} \\ \vdots & \vdots & \vdots \\ \sigma_{\tau 1} \cdots & \sigma^2_{\delta_\tau} \cdots & \sigma_{\tau t} \\ \vdots & \vdots & \vdots \\ \sigma_{t1} \cdots & \sigma_{t\tau} \cdots & \sigma^2_{\delta_t} \end{bmatrix} \tag{5}$$

where the elements of $\underline{\underline{\Sigma}}$ are determined as follows: defining for $1 \leq \ell \leq L$ and $i=1,2,\ldots,N$,[†]

$$c_{\tau,\tau+\ell} = E(\tilde{y}_{i,\tau+\ell}\ \tilde{\epsilon}_{i\tau}) \tag{6a}$$

and

$$q_{\tau,\tau+\ell} = P\left(\sum_{t=\tau+\ell-L}^{\tau+\ell-1} \tilde{\epsilon}_{it} > X_{\tau+\ell}\ ,\ \sum_{t=\tau-L}^{\tau-1} \tilde{\epsilon}_{it} > X_\tau\right)\ , \tag{6b}$$

we have

$$\sigma_{\tau\tau'} = \begin{cases} 0 & \text{if } \tau \neq \tau'+\ell \\ r^2 N[q_{\tau,\tau+\ell} - p_\tau p_{\tau+\ell}] - rkNc_{\tau,\tau+\ell} & \tau = \tau'+\ell \end{cases} \tag{6c}$$

and

$$\sigma^2_{\delta_\tau} = N\{r^2 p_\tau(1-p_\tau) + k^2\sigma^2_\epsilon\}\ , \quad \tau=1,2,\ldots,t. \tag{7}$$

From (2), (5), (6a), (6b), (6c), and (7) it follows that

$$E(\tilde{U}_t) = U_0 - \alpha t + kf(t) - rN \sum_{\tau=1}^{t} p_\tau\ , \tag{8}$$

$$V(\tilde{U}_t) = V\left(\sum_{\tau=1}^{t} \tilde{\delta}_\tau\right) = \sum_{\tau=1}^{t} \sigma_{\delta_\tau} + 2 \sum_{\tau=1}^{t-1} \sum_{\tau'>\tau} \sigma_{\tau\tau'}\ . \tag{9}$$

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[†]See formula (10) for a definition of the random variables $\tilde{y}_{i,\tau+\ell}$.

We may calculate both the mean and variance of the number of workers covered per time period as functions of p_τ, $\tau=1,2,\ldots$ and $q_{\tau,\tau+\ell}$, $1 \le \ell \le L$, and so facilitate analysis of their behavior as functions of the parameter (X,L) of the eligibility rule: Define

$$\eta_t = \frac{1}{t} \sum_{\tau=1}^{t} \tilde{n}_\tau$$ - average number of members covered per time period over t periods of time.

We show below that

$$E(\tilde{\eta}_t) = \frac{N}{t} \sum_{\tau=1}^{t} p_\tau \tag{10a}$$

and

$$V(\tilde{\eta}_t) = \frac{N}{t^2} \sum_{\tau=1}^{t} p_\tau(1-p_\tau) + \frac{2N}{t^2} \sum_{\tau=1}^{t-1} \sum_{\tau'>\tau}^{t} (q_{\tau\tau'} - p_\tau p_{\tau'}) \tag{10b}$$

PROOFS: Before proving (4) through (10), observe that the data generating process which gives n_t may be thought of as a Bernoulli process: define for given L

$$y_{i,t+1} = \begin{cases} 1 & \text{if} \quad \sum_{\tau=t-L+1}^{t} h_{i\tau} \geq X \\ 0 & \quad\;\; \sum_{\tau=t-L+1}^{t} h_{i\tau} < X \end{cases} \qquad \text{all } i, \text{ all } t. \tag{11}$$

If we define

$$X_t = X - f(t) \text{ and } s_{it} = \sum_{\tau=t-L+1}^{t} \epsilon_{i\tau} \quad ,$$

then we may write (11) as

$$y_{i,t+1} = \begin{cases} 1 & s_{i\tau} \geq X_t \\ 0 & s_{it} < X_t \end{cases} \text{, all } i, \text{ all } t,$$

by virtue of II.

Since the $\tilde{\epsilon}_{it}$s are mutually independent by Assumption II, so are the random variables $\tilde{y}_{it}$, and furthermore, since

$$P(\tilde{s}_{it} \geq X_t) = P(\tilde{s}_{jt} \geq X_t) \equiv p_t \quad , \quad \text{all } 1 \leq i, j \leq N, \text{ all } t, \tag{12}$$

for a <u>given</u> t we may regard $y_{1t}, y_{2t}, \ldots, y_{it}, \ldots, y_{Nt}$ as values generated by a Bernoulli process with parameter p_t.

It will be convenient to work with the y_{it}s rather than n_t in the subsequent proofs:

Proof of (4): From (1) and (3) we have for $t \geq 1$,

$$\tilde{\delta}_t = [U_0 - \alpha t + f(t) + \sum_{\tau=1}^{t} \sum_{i=1}^{N} (k\,\tilde{\epsilon}_{i\tau} - r\tilde{y}_{i\tau})]$$

$$- [U_0 - \alpha(t-1) + f(t-1) + \sum_{\tau=1}^{t-1} \sum_{i=1}^{N} (k\,\tilde{\epsilon}_{i\tau} - r\tilde{y}_{i\tau})]$$

$$= -\alpha + f(t) - f(t-1) + \sum_{i=1}^{N} (k\,\tilde{\epsilon}_{it} - r\tilde{y}_{it}) .$$

Using II and the fact that $E(\tilde{y}_{it}) = p_t$, all i,

$$\bar{\delta}_t = -\alpha + f(t) - f(t-1) - rNp_t .$$

Proof of (7): Assumption II implies that $\tilde{y}_{i\tau}$ is independent of $\tilde{y}_{j\tau}$, $j \neq i$, and also that $\tilde{y}_{i\tau}$ is independent of $\tilde{\epsilon}_{j\tau}$, $j \neq i$. Thus, using II and (12),

$$V(\tilde{\delta}_\tau) = V(\sum_{i=1}^{N} (k\tilde{\epsilon}_{i\tau} - r\tilde{y}_{i\tau})) = \sum_{i=1}^{N} k^2 V(\tilde{\epsilon}_{i\tau}) + r^2 V(\tilde{y}_{i\tau})$$

$$= Nr^2 p_\tau(1-p_\tau) + Nk^2\sigma_\epsilon^2 , \quad \tau=1,2,\ldots,t.$$

for Assumptions I and II together imply that $\tilde{y}_{i\tau}$ and $\tilde{\epsilon}_{i\tau}$ are independent.

Proof of (6): To establish (6) observe that I and II imply that

(a) when $\tau \neq \tau' + \ell$, $\ell=1,2,\ldots,L$, $\tilde{y}_{i\tau}$ is independent of $\tilde{\epsilon}_{j\tau'}$ for all i and j;

(b) when $\tau = \tau' + \ell$ for some ℓ, $1 \leq \ell \leq L$, $\tilde{y}_{i\tau}$ is independent of $\tilde{\epsilon}_{j\tau'}$ only when $i \neq j$, but $\tilde{y}_{i\tau}$ and $\tilde{\epsilon}_{i\tau'}$ are correlated;

(c) when $i \neq j$, $\tilde{y}_{i\tau}$ and $\tilde{y}_{j\tau'}$ are independent for all τ and τ';

(d) when $i=j$ and $\tau \neq \tau'$, $\tilde{y}_{i\tau}$ and $\tilde{y}_{j\tau'}$ are correlated if $\tau=\tau'+\ell-1$ and $1 \leq \ell \leq L$, and are independent otherwise.

An example will help to clarify the meaning of (a), (b), and (d). Suppose $L = 3$ and we display $\tilde{y}_{i\tau}$ and $\tilde{\epsilon}_{i\tau'}$ as shown below for $\tau'=1,2,\ldots$ An x in the (τ')th column and τth row indicates that $\tilde{y}_{i\tau}$ and $\tilde{\epsilon}_{i\tau'}$ are correlated.

	τ':	1	2	3	4	5	6	7	.	.	.
τ		ϵ_{i1}	ϵ_{i2}	ϵ_{i3}	ϵ_{i4}	ϵ_{i5}	ϵ_{i6}	ϵ_{i7}	.	.	.
2	y_{i2}	x									
3	y_{i3}	x	x								
4	y_{i4}	x	x	x							
5	y_{i5}		x	x	x						
6	y_{i6}			x	x	x					
7	y_{i7}				x	x	x				
8	y_{i8}					x	x	x			
	.						.	. .	.	.	
	.							.	.	.	
	.								.	.	.

For example, when $\tau \neq \tau' + 1,\ \tau' + 2,\ \tau' + 3$, $\tilde{y}_{i\tau}$ and $\tilde{\epsilon}_{i\tau'}$ are uncorrelated, while if, say, $\tau = \tau' + 2$, $\tilde{y}_{i\tau}$ and $\tilde{\epsilon}_{i\tau'}$ are correlated. Furthermore since $\tilde{y}_{i,\tau'+3}$, say, is correlated with $\tilde{\epsilon}_{i,\tau'+2}$, $\tilde{\epsilon}_{i,\tau'+1}$, and $\tilde{\epsilon}_{i,\tau'}$, and $\tilde{y}_{i,\tau'+4}$ is correlated with $\tilde{\epsilon}_{i,\tau'+3}$, $\tilde{\epsilon}_{i,\tau'+2}$, $\tilde{\epsilon}_{i,\tau'+1}$, then $\tilde{y}_{i,\tau'+4}$ and $\tilde{y}_{i,\tau'+3}$ are correlated. However, $\tilde{y}_{i,\tau'+4}$ is uncorrelated with $\tilde{y}_{i,\tau'+1}$.

<u>Case I: $\tau=\tau'+\ell, \quad 1 \leq \ell \leq L$</u>

1. First we show that we must evaluate the term $E(\tilde{\delta}_\tau \tilde{\delta}_{\tau+\ell})$ In order to prove (6) when $\tau=\tau'+\ell$ and $1 \leq \ell \leq L$:

$$\text{Cov}(\tilde{\delta}_\tau \tilde{\delta}_{\tau+\ell}) = E([\tilde{\delta}_\tau - \bar{\delta}_\tau][\tilde{\delta}_{\tau+\ell} - \bar{\delta}_{\tau+\ell}])$$

$$= E(\tilde{\delta}_\tau \tilde{\delta}_{\tau+\ell}) - \bar{\delta}_\tau \bar{\delta}_{\tau+\ell} \quad .$$

We now show that for all i

$$\text{Cov}(\tilde{y}_{i,\tau+\ell}, \tilde{\epsilon}_{i\tau}) = E(\tilde{y}_{i,\tau+\ell} \tilde{\epsilon}_{i\tau})$$

$$= E_b E^{\infty}_{X_{\tau+\ell}-b} (\tilde{\epsilon}_{i\tau} | \tilde{b}) = c_{\tau,\tau+\ell}$$

where $\tilde{b}$ is defined in (15) below and that

$$\text{Cov}(\tilde{y}_{i,\tau+\ell}, \tilde{y}_{i\tau}) = E(\tilde{y}_{i,\tau+\ell} \tilde{y}_{i\tau}) - p_{\tau+\ell} p_\tau$$

$$= q_{\tau,\tau+\ell} - p_\tau p_{\tau+\ell} \, ,$$

for we need these results to evaluate (14). We first evaluate $E(\tilde{y}_{i,\tau+\ell} \tilde{\epsilon}_{i\tau})$. Remember that

$$y_{i,\tau+\ell} = \begin{cases} 1 & \text{if } \sum\limits_{\tau'=\tau+\ell-L}^{\tau+\ell-1} \epsilon_{i\tau'} \geq X_{\tau+\ell} \\ 0 & \phantom{\text{if }} \sum\limits_{\tau'=\tau+\ell-L}^{\tau+\ell-1} \epsilon_{i\tau'} < X_{\tau+\ell} \end{cases} \quad .$$

We have assumed that $\tau+\ell-1 \geq \tau' \geq \tau+\ell-L$, so for notational convenience define

$$b = \left(\sum_{\tau'=\tau+\ell-L}^{\tau+\ell-1} \epsilon_{i\tau'} \right) - \epsilon_{i\tau}, \quad \text{all i.}$$

Now given $\tilde{b} = b$, and $\tilde{\epsilon}_{i\tau} = z$,

$$P(\tilde{y}_{i,\tau+\ell} = 1 \mid b, z) = \begin{cases} 1 & \text{if} \quad b+z \geq X_{\tau+\ell} \\ 0 & \phantom{\text{if} \quad b+z} < X_{\tau+\ell} \end{cases}$$

and

$$P(\tilde{y}_{i,\tau+\ell} = 0 \mid b, z) = \begin{cases} 0 & \text{if} \quad b+z \geq X_{\tau+\ell} \\ 1 & \phantom{\text{if} \quad b+z} < X_{\tau+\ell} \end{cases}$$

Defining for all i and τ,

$$F_\epsilon(z) = P(\tilde{\epsilon}_{i\tau} < z) \text{ and } F_b(b) = P(\tilde{b} < b)$$

we have

$$E(\tilde{y}_{i,\tau+\ell}\, \tilde{\epsilon}_{i\tau}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [\sum_{y=0}^{1} yzP(y|z,u)]\ dF_{\epsilon}(z)dF_b(u)$$

$$= \int_{(X_{\tau+\ell}-u)}^{\infty} \int_{-\infty}^{\infty} z\ dF_\epsilon(z)dF_b(u) \tag{15a}$$

$$= E_b E^{\infty}_{X_{\tau+\ell}-b} (\tilde{\epsilon}_{i\tau}|\tilde{b}) = c_{\tau,\tau+\ell} \ .$$

This last formula will be used to evaluate $c_{\tau,\tau+\ell}$ numerically.

We may evaluate $E(\tilde{y}_{i,\tau+\ell}\tilde{y}_{i\tau})$ in a similar fashion. Define

$$\tilde{V}_{i\tau} = \sum_{t=\tau+\ell-L}^{\tau-1} \tilde{\epsilon}_{it} \quad . \tag{15b}$$

$$\tilde{W}_{i,\tau+\ell} = \sum_{t=\tau}^{\tau+\ell-1} \tilde{\epsilon}_{it} \quad , \tag{15c}$$

and

$$\tilde{\Gamma}_{i\tau} = \sum_{t=\tau-L}^{\tau+\ell-L-1} \tilde{\epsilon}_{it} \quad , \tag{15d}$$

so that

$$\tilde{y}_{i\tau} = \begin{cases} 1 & \text{if } \tilde{\Gamma}_{i\tau} + \tilde{V}_{i\tau} \geq X_\tau \\ 0 & \qquad\qquad\qquad < X_\tau \end{cases}$$

and

$$\tilde{y}_{i,\tau+\ell} = \begin{cases} 1 & \text{if } \tilde{V}_{i\tau} + \tilde{W}_{i,\tau+\ell} \geq X_{\tau+\ell} \\ 0 & \qquad\qquad\qquad < X_{\tau+\ell} \end{cases}$$

Hence if $\tilde{\Gamma}_{i\tau} = \Gamma$, $\tilde{V}_{i\tau} = V$, and $\tilde{W}_{i,\tau+\ell} = W$, then

$$P(\tilde{y}_{i\tau} = 1|\Gamma,V) = \begin{cases} 1 & \text{if } \Gamma_{i\tau} + V_{i\tau} \geq X_\tau \\ 0 & \qquad\qquad\qquad < X_\tau \end{cases}$$

$$P(\tilde{y}_{i,\tau+\ell} = 1|\Gamma, W) = \begin{cases} 1 & \text{if } V_{i\tau} + W_{i,\tau+\ell} \geq X_{\tau+\ell} \\ 0 & \qquad\qquad\qquad < X_{\tau+\ell} \end{cases}$$

Define for all i and τ, and for $1 \leq \ell \leq L$

$$F_{X_\tau}(v) = P(\tilde{V}_{i\tau} < v),$$

$$F_{W_{\tau+\ell}}(w) = 1 - G_{W_{\tau+\ell}}(w) = P(\tilde{W}_{i,\tau+\ell} < w) \ ,$$

and

$$F_{\Gamma_\tau}(g) = 1 - G_{\Gamma_\tau}(g) = P(\tilde{\Gamma}_{i\tau} < g) \quad .$$

Thus using the fact that $\tilde{V}_{i\tau}$, $\tilde{W}_{i,\tau+\ell}$, and $\tilde{\Gamma}_{i\tau}$ are mutually independent, since the $\tilde{\epsilon}_{i\tau}$'s are mutually independent for all i and τ, [illegible] hav[illegible] [illegible] a formula for calculating $q_{\tau,\tau+\ell}$ when $1 \leq \ell \leq L$:

$$E(\tilde{y}_{i,\tau+\ell}\,\tilde{y}_{i\tau}) = \int_{-\infty}^{\infty} G_{\Gamma_\tau}(X_\tau - v)\, G_{W_{\tau+\ell}}(X_{\tau+\ell} - v)\, dF_{V_\tau}(v) \equiv q_{\tau,\tau+\ell} . \tag{15e}$$

3. We evaluate $E(\tilde{\delta}_\tau \tilde{\delta}_{\tau+\ell})$ by noting that from (3),

$$\tilde{\delta}_\tau \tilde{\delta}_{\tau+\ell} = (\zeta_\tau + \sum_{i=1}^{N} (k\tilde{\epsilon}_{i\tau} - r\tilde{y}_{i\tau})\ (\zeta_{\tau+\ell} + \sum_{i=1}^{N} (k\tilde{\epsilon}_{i,\tau+\ell} - r\tilde{y}_{i,\tau+\ell})\ )$$

$$= \zeta_{\tau+\ell}(\sum_{i=1}^{N} k\tilde{\epsilon}_{i\tau} - r\tilde{y}_{i\tau}) + \zeta_\tau(\sum_{i=1}^{N} k\tilde{\epsilon}_{i,\tau+\ell} - r\tilde{y}_{i,\tau+\ell}) + \zeta_\tau\zeta_{\tau+\ell}$$

$$+ (\sum_{i=1}^{N} k\tilde{\epsilon}_{i\tau} - r\tilde{y}_{i\tau})\ (\sum_{i=1}^{N} k\tilde{\epsilon}_{i,\tau+\ell} - r\tilde{y}_{i,\tau+\ell})$$

where

$$\zeta_\tau = -\alpha + f(t) - f(t-1) \quad ,$$

$$\zeta_{\tau+\ell} = -\alpha + f(\tau+\ell) - f(t+\ell-1) \quad ,$$

so that using (12) and II,

$$E(\tilde{\delta}_\tau \tilde{\delta}_{\tau+\ell}) = (\zeta_\tau \zeta_{\tau+\ell} - rN\zeta_{\tau+\ell} p_\tau - rN\zeta_\tau p_{\tau+\ell})$$

$$+ E\left(\sum_{i=1}^{N} (k\tilde{\epsilon}_{i\tau} - r\tilde{y}_{i\tau}) \sum_{j=1}^{N} (k\tilde{\epsilon}_{j,\tau+\ell} - r\tilde{y}_{j,\tau+\ell}) \right).$$

To finish proof of Case I we show that

$$E\left(\left[\sum_{i=1}^{N} (k\tilde{\epsilon}_{i\tau} - r\tilde{y}_{i\tau})\right] \left[\sum_{j=1}^{N} (k\tilde{\epsilon}_{j,\tau+\ell} - r\tilde{y}_{j,\tau+\ell})\right]\right)$$

$$= r^2 N q_{\tau,\tau+\ell} + r^2 N(N-1) p_\tau p_{\tau+\ell} - kNc_{\tau,\tau+\ell}.$$

By virtue of (a), (b), (c), and (d) stated at the outset of the proof this reduces to

$$r^2 \sum_{i=1}^{N} E(\tilde{y}_{i\tau} y_{i,\tau+\ell}) + r^2 N(N-1) p_\tau p_{\tau+\ell}$$

$$-rk \sum_{j=1}^{N} E(\tilde{y}_{j,\tau+\ell} \tilde{\epsilon}_{j\tau}).$$

Using the definitions of $q_{\tau,\tau+\ell}$, p_τ and $c_{\tau,\tau+\ell}$ the above may be written as

$$r^2 N q_{\tau,\tau+\ell} + r^2 N(N-1)\, p_\tau p_{\tau+\ell} - rkNc_{\tau,\tau+\ell} \quad .$$

From (14), (16), and (17), we have

$$\mathrm{Cov}(\tilde{\delta}_\tau, \tilde{\delta}_{\tau+\ell}) = (\zeta_\tau \zeta_{\tau+\ell}\ rN\zeta_{\tau+\ell} p_\tau - rN\zeta_\tau p_{\tau+\ell})$$

$$+ (r^2 N q_{\tau,\tau+\ell} + r^2 N(N-1) p_\tau p_{\tau+\ell} - rkNc_{\tau,\tau+\ell})$$

$$- \bar{\delta}_\tau \bar{\delta}_{\tau+\ell} \quad .$$

Since

$$\bar{\delta}_\tau \bar{\delta}_{\tau+\ell} = (\zeta_\tau - Np_\tau)(\zeta_{\tau+\ell} - Np_{\tau+\ell})$$

$$= \zeta_\tau \zeta_{\tau+\ell} - rNp_\tau \zeta_{\tau+\ell} - rNp_{\tau+\ell}\zeta_\tau + r^2N^2 p_\tau p_{\tau+\ell} \quad ,$$

$$\mathrm{Cov}(\tilde{\delta}_\tau, \tilde{\delta}_{\tau+\ell}) = r^2 Nq_{\tau,\tau+\ell} - r^2 Np_\tau p_{\tau+\ell} - rkNc_{\tau,\tau+\ell}$$

which completes the proof for the case $\tau = \tau' + \ell, \quad 1 \leq \ell \leq L$.

Case II: $\tau \neq \tau' + \ell,\ 1 \leq \ell \leq L,$

If $\tau \neq \tau' + \ell$ we have

$$\mathrm{Cov}(\tilde{\delta}_\tau, \tilde{\delta}_{\tau'}) = E(\tilde{\delta}_\tau \tilde{\delta}_{\tau'}) - \bar{\delta}_\tau \bar{\delta}_{\tau'}$$

as before, and by analogy with (16) we have

$$E(\tilde{\delta}_\tau \tilde{\delta}_{\tau'}) = \zeta_\tau \zeta_{\tau'} - rNp_\tau \zeta_{\tau'} - rNp_{\tau'}\zeta$$

$$+ E([\sum_{i=1}^{N} (k\tilde{\epsilon}_{i\tau} - r\tilde{y}_{i\tau})]\ ]\ \sum_{j=1}^{N} (k\tilde{\epsilon}_{j\tau'} - r\tilde{y}_{j\tau'})]) \quad .$$

The expectation on the right hand side above reduces to $r^2N^2 p_\tau p_{\tau'}$ by use of (a), (b), and (c), giving

$$\mathrm{Cov}(\tilde{\delta}_\tau, \tilde{\delta}_{\tau'}) = \zeta_\tau \zeta_{\tau'} - rN(p_\tau \zeta_\tau - rNp_\tau p_{\tau'}) - \bar{\delta}_\tau \bar{\delta}_{\tau'} \ ,$$

but since

$$\bar{\delta}_\tau \bar{\delta}_{\tau'} = (\zeta_\tau - rNp_\tau)\,(\zeta_{\tau'} - rNp_{\tau'})$$

$$= \zeta_\tau \zeta_{\tau'} - rN(p_\tau \zeta_{\tau'} + p_{\tau'} \zeta_\tau - rNp_\tau p_{\tau'}) \quad ,$$

$$\mathrm{Cov}(\tilde{\delta}_\tau, \tilde{\delta}_{\tau'}) = 0 \quad , \qquad \tau \neq \tau' + \ell \;, \qquad 1 \leq \ell \leq L.$$

<u>Proof of (10)</u>: The proof of (10a) and (10b) is in fact imbedded in that of (9), for from the proof of (9) we have

$$V(\tilde{\eta}_t) = V\left(\frac{1}{t} \sum_{\tau=1}^{t} \tilde{n}_\tau\right) = \frac{1}{t^2} V\left(\sum_{\tau=1}^{t} \tilde{n}_\tau\right)$$

$$= \frac{1}{t^2} \sum_{\tau=1}^{t} V(\tilde{n}_\tau) + \frac{2}{t^2} \sum_{\tau=1}^{t-1} \sum_{\tau'>\tau}^{t} \mathrm{Cov}(\tilde{n}_\tau, \tilde{n}_{\tau'})$$

$$= \frac{N}{t^2} \sum_{\tau=1}^{t} p_\tau (1-p_\tau) + \frac{2N}{t^2} \sum_{\tau=1}^{t-1} \sum_{\tau'>\tau}^{t} (q_{\tau\tau'} - p_\tau p_{\tau'})$$

Formula (10a) is obvious.

2.2 Asymptotic Normality of $\tilde{\underline{\delta}}_t$

The following Lemma is of considerable practical importance because it allows us to simulate a sample realization $\{\delta_1, \delta_2, \ldots, \delta_t\}$ by generating only t random Normal deviates when N is large, in place of generating 2Nt random numbers: N ys and N εs for each of t δs.

Lemma 1: As $N \to \infty$ the random vector $\tilde{\underline{\delta}}_t$ is asymptotically distributed as a multivariate Normal vector with mean $\bar{\underline{\delta}}_t$ and covariance matrix $\underline{\underline{\Sigma}}$.

PROOF:

Define the 2t x 1 vector

$$\underline{x}_i = (h_{i1}, \ldots, h_{it}, y_{i1}, \ldots, y_{it})^t \tag{18}$$

and the (t x 2t) matrix

$$\underline{\underline{A}} = \begin{bmatrix} k & 0 & 0 & 0 & \ldots & 0 & r & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & k & 0 & 0 & \ldots & 0 & 0 & r & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & k & 0 & \ldots & 0 & 0 & 0 & r & 0 & \ldots & 0 & 0 & 0 \\ & & & \cdot & & & & \cdot & & & & & \cdot & \\ & & & \cdot & & & & \cdot & & & & & \cdot & \\ & & & \cdot & & & & \cdot & & & & & \cdot & \\ 0 & 0 & 0 & 0 & \ldots & k & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & r \end{bmatrix} \tag{19}$$

so that

$$\sum_{i=1}^{N} \underline{\underline{A}}\, \underline{x}_i = \underline{\delta}_t \quad . \tag{20}$$

If we consider $\underline{\tilde{x}}_i$ as a random vector, observe that Assumption II implies that the $\{\underline{\tilde{x}}_i,\ i=1,2,\ldots,N\}$ are mutually independent and identically distributed with a mean $\underline{\bar{x}}$ and a symmetric covariance matrix $\underline{\underline{T}}$.

We may now use a multivariate central limit theorem* to prove that the random vector $\underline{\tilde{\delta}}_t$ is asymptotically multivariate Normal with mean vector $N\, \underline{\underline{A}}\, \underline{\tilde{x}}$ and covariance matrix $N\, \underline{\underline{A}}\, \underline{\underline{T}}\, \underline{\underline{A}}^t$ as $N \to \infty$:

<u>Theorem</u>:

Let the 2t component random vectors $\underline{\tilde{x}}_i$ be independent and identically distributed with means $\underline{\bar{x}}$ and covariance matrices $E(\underline{\tilde{x}}_i - \underline{\tilde{x}})(\underline{\tilde{x}}_i - \underline{\tilde{x}})^t \equiv \underline{\underline{T}}$. Then the limiting distribution of $\underline{\tilde{z}} = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (\underline{\tilde{x}}_i - \underline{\bar{x}})$ as $N \to \infty$ is

$$f_N^{(2t)}(\underline{z}\,|\,\underline{0},\underline{\underline{T}}) = (2\pi)^{-\frac{t}{2}}\, e^{-\frac{1}{2}\underline{z}^t (\underline{\underline{T}})^{-1} \underline{z}}\, |\underline{\underline{T}}|^{-1} \quad .$$

The theorem thus implies that as $N \to \infty$,

$$\underline{\tilde{X}} \equiv \sum_{i=1}^{N} \underline{\tilde{x}}_i = \sqrt{N}\tilde{z} + N\underline{\tilde{x}} \sim f_N^{(2t)}(\underline{X}_i\,|\,N\underline{x}, N\underline{\underline{T}})$$

or

$$\underline{\tilde{\delta}}_t = \sum_{i=1}^{N} \underline{\underline{A}}\, \underline{x}_i \sim f_N^{(t)}(\underline{\tilde{\delta}}_t\,|\,N\underline{\underline{A}}\underline{\tilde{x}}\ ,\ N\underline{\underline{A}}\,\underline{\underline{T}}\ \underline{\underline{A}}^t).$$

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* See T. W. Anderson, <u>Introduction to Multivariate Statistical Analysis</u> (John Wiley & Sons, New York, 1958).

2.3 Structure of the Covariance Matrices $\underline{\underline{T}}$ and $\underline{\underline{\Sigma}}$

The covariance matrix $\underline{\underline{T}}$ of $\underline{\tilde{x}}_i$, all i, may be written as

$$\underline{\underline{T}} = \begin{bmatrix} \beta \underline{\underline{I}} & \underline{\underline{H}} \\ \underline{\underline{H}}^t & \underline{\underline{Y}} \end{bmatrix} \tag{21a}$$

where $\beta = \sigma_\epsilon^2$, the matrix $\underline{\underline{I}}$ is a (t x t) identity matrix, and the (t x t) matrices $\underline{\underline{H}}$ and $\underline{\underline{Y}}$ are defined below.

Before we define $\underline{\underline{H}}$ and $\underline{\underline{Y}}$ explicitly, observe that since the matrix $\underline{\underline{A}}$ of (19) may be written as $(k\,\underline{\underline{I}} \quad r\,\underline{\underline{I}})$, where $\underline{\underline{I}}$ is identity matrix of order t, and since $\underline{\underline{\Sigma}} = \underline{\underline{A}}\,\underline{\underline{T}}\,\underline{\underline{A}}^t$, we may use (2.1) to represent $\underline{\underline{\Sigma}}$ in the following convenient form:

$$\underline{\underline{\Sigma}} = \beta k^2 \underline{\underline{I}} + 2rk\underline{\underline{\Phi}} + r^2\underline{\underline{Y}} \tag{21b}$$

where

$$\underline{\underline{\Phi}} \equiv \underline{\underline{H}}^t + \underline{\underline{H}} \quad ,$$

for

$$\underline{\underline{\Sigma}} = \underline{\underline{A}}\,\underline{\underline{T}}\,\underline{\underline{A}}^t = (k\,\underline{\underline{I}} \quad r\,\underline{\underline{I}}) \begin{bmatrix} \beta\underline{\underline{I}} & \underline{\underline{H}} \\ \underline{\underline{H}}^t & \underline{\underline{Y}} \end{bmatrix} \begin{pmatrix} k\underline{\underline{I}} \\ r\underline{\underline{I}} \end{pmatrix}$$

$$= \beta k^2 \underline{\underline{I}} + rk(\underline{\underline{H}}^t + \underline{\underline{H}}) + r^2\underline{\underline{Y}} \quad .$$

Both $\underline{\underline{H}}$ and $\underline{\underline{Y}}$ are determined by the structure of the eligibility rule. Given an eligibility rule as stated in Assumption I, a worker must work a total X hours or more during periods t-1,t-2,...t-L to be covered in period t, it is clear that $\tilde{y}_{i\tau}$ will be correlated with $\tilde{h}_{i\tau'}$ for τ' = t-1,t-2,...t-L and with $\tilde{y}_{i\tau'}$ for τ' = t-1,t-2,...,t-L+1 <u>and</u> for τ' = t+1, t+2,..., t+L-1. We may conveniently display the pairwise covariances of a sequence $\{\tilde{y}_{i\tau},$ τ=1,2,...,t$\}$ in covariance matrix form:

$$\underline{\underline{Y}} = \left[\begin{array}{ccccccccc}
Y_{11} & \cdots & Y_{1L} & & & & & & \\
\vdots & & \vdots & & & & \underline{\underline{0}} & & \\
Y_{L1} & \cdots & Y_{LL} & \cdots & Y_{L,2L-1} & & & & \\
& \ddots & & \ddots & & \ddots & & & \\
& \underline{\underline{0}} & & & Y_{t-L+1,t-2L+2} & \cdots & Y_{t-L+1,t-L+1} & \cdots & Y_{t-L+1,t} \\
& & & & & & \vdots & & \vdots \\
& & & & & & Y_{t,t-L+1} & \cdots & Y_{tt}
\end{array}\right]$$

where for all i, $\mathrm{Cov}(\tilde{y}_{i\tau'}, \tilde{y}_{\tau'\tau}) = Y_{\tau'\tau} = q_{\tau\tau'} - p_{\tau}p_{\tau'}$. We may display the covariances of the ys and hs in a similar fashion as:

$$\begin{bmatrix} 0 & H_{12} & H_{13} & \cdots & H_{1,L+1} & & & \\ & 0 & H_{23} & \cdots & & H_{2,L+2} & \underline{\underline{0}} & \\ & & \ddots & \ddots & & \ddots & & \\ & & & \ddots & \ddots & & \ddots & \\ & & & & & & & H_{t-L,t} \\ & \underline{\underline{0}} & & & & \ddots & \ddots & \vdots \\ & & & & & & 0 & H_{t-1,t} \\ & & & & & & & 0 \end{bmatrix}$$

where for $\tau = 1,2,\ldots,t,$

$$\mathrm{Cov}(\tilde{h}_{i\tau'}, \tilde{y}_{i\tau}) = \begin{cases} H_{\tau'\tau} = E(\tilde{\epsilon}_{i\tau'}, \tilde{y}_{i\tau}) = c_{\tau'\tau} & \text{for } \tau=\tau'+\ell,\ 1 \le \ell \le L \\ 0 & \text{otherwise} \end{cases} .$$

2.4 Calculation of $q_{\tau,\tau+\ell}$

If we make Assumption IV:

> $\{\tilde{\epsilon}_{i\tau},\ i=1,2,\ldots,N;\ \tau=1,2,\ldots,t\}$ is a double sequence of mutually independent identically distributed Normal random variables, each with mean 0 and variance σ_ϵ^2,

then for each i and for $1 \leq \ell \leq L$, we have from (15b) and (15d)

$$\mathrm{Var}(\tilde{V}_{i\tau}) = (L-\ell)\sigma_\epsilon^2 ,$$

$$\mathrm{Var}(\tilde{W}_{i,\tau+\ell}) = \ell\, \sigma_\epsilon^2 ,$$

$$\mathrm{Var}\ (\tilde{\Gamma}_{i\tau}) = \ell\, \sigma_\epsilon^2 .$$

It follows that the (3 x 1) random vector

$$\tilde{\underline{\zeta}}^t_{i\tau} \equiv (\tilde{V}_{i\tau}\ \ \tilde{W}_{i,\tau+\ell}\ \ \tilde{\Gamma}_{i\tau})$$

is Multinormal with mean vector $(0\ 0\ 0)^t$ and covariance matrix

$$\sigma_\epsilon^2 \begin{bmatrix} L-\ell & 0 & 0 \\ 0 & \ell & 0 \\ 0 & 0 & \ell \end{bmatrix} .$$

Hence for all i the (2 x 1) random vector

$$\underline{R}^t = (\tilde{R}_\tau\ \tilde{R}_{\tau+\ell}) \equiv (\tilde{\Gamma}_{i\tau} + \tilde{V}_{i\tau}\ \ \ \tilde{V}_{i\tau} + \tilde{W}_{i,\tau+\ell})^t$$

is Multinormal with mean vector $(0\ 0^t)$ and covariance matrix

$$\sigma_\epsilon^2 \begin{bmatrix} L & L-\ell \\ & \\ L-\ell & L \end{bmatrix} = \frac{\sigma_\epsilon^2}{L} \begin{bmatrix} 1 & (1-\frac{\ell}{L}) \\ & \\ (1-\frac{\ell}{L}) & 1 \end{bmatrix} . \tag{22}$$

The probability

$$P(\tilde{R}_T > X_T, \tilde{R}_{T+\ell} > X_{T+\ell}) = q_{T,T+\ell}$$

may be looked up in tables of the cumulative unit-elliptical bivariate Normal function

$$L(h,k,r) \equiv \int_h^\infty \int_k^\infty f_N^{(2)}(\underline{z}|\underline{0}, \underline{\underline{\Sigma}})\, dz_1\, dz_2 \tag{23}$$

where $-1 \leq r \leq 1$ and

$$\underline{\underline{\Sigma}} = \begin{bmatrix} 1 & r \\ r & 1 \end{bmatrix} ,$$

tabulated by the National Bureau of Standards: <u>Tables of the Bivariate Normal Distribution Function and Related Functions</u>, Applied Mathematics Series 50, U.S.G.P.O., Washington D.C., 1959.

Notice that $q_{T,T+\ell}$ is almost in proper form for table lookup. We need only observe that

$$P(\tilde{R}_T > X_T, \tilde{R}_{T+\ell} > X_{T+\ell}) = P(\tilde{z}_T > z_T, \tilde{z}_{T+\ell} > z_{T+\ell})$$

where

$$\tilde{z}_T = \sqrt{L}\tilde{R}_T/\sigma_\epsilon , \quad z_T = \sqrt{L}X_T/\sigma_\epsilon , \quad \tilde{z}_{T+\ell} = \sqrt{L}\tilde{R}_{T+\ell}/\sigma_\epsilon , \quad z_{T+\ell} = \sqrt{L}X_{T+\ell}/\sigma_\epsilon ;$$

then from (23)

$$P(\tilde{R}_T > X_T, \tilde{R}_{T+\ell} > X_{T+\ell}) = L\left(\frac{\sqrt{L}X_T}{\sigma_\epsilon}, \frac{\sqrt{L}X_{T+\ell}}{\sigma_\epsilon}, 1 - \frac{\ell}{L}\right) .$$

However, we shall use the computer to generate values of $q_{T,T+\ell}$ needed in the course of the simulation.

2.5 Calculation of $c_{\tau,\tau+\ell}$

We show below that

$$c_{\tau,\tau+\ell} \equiv E(\tilde{y}_{i,\tau+\ell}\tilde{\epsilon}_{i\tau}) = f_{N*}(\sqrt{H}\, X_{\tau+\ell}) \quad \text{where} \quad H = (L\sigma_\epsilon^2)^{-1} \tag{24}$$

if we make Assumption IV stated in 2.4: For $i=1,2,\ldots,N$ and $\tau=1,2,\ldots,t$ the $\tilde{\epsilon}_{i\tau}$ are mutually independent identically distributed Normal random variables with mean 0 and variance σ_ϵ^2.

Proof: From (15a) we have

$$c_{\tau,\tau+\ell} = \int_{(X_{\tau+\ell}-u)}^{\infty} \int_{-\infty}^{\infty} z \, dF_{\epsilon_\tau}(z) \, dF_b(u)$$

If

$$F_\epsilon(z) = F_N(z|0,h_\epsilon)$$

and

$$F_b(u) = F_N(u|0,h_b)$$

then

$$c_{\tau,\tau+\ell} = \int_{(X_{\tau+\ell}-u)}^{\infty} \int_{-\infty}^{\infty} z f_N(z|0,h_\epsilon) \, f_N(u|0,h_b) \, dz du \quad .$$

Since

$$\int_{(X_{\tau+\ell}-u)}^{\infty} z \, f_N(z|0,h_z) dz$$

$$= \int_{\sqrt{h_\epsilon}(X_{\tau+\ell}-u)}^{\infty} v \, f_{N*}(v) \, dv = + f_{N*}(\sqrt{h_\epsilon}(X_{\tau+\ell}-u))$$

we have

$$c_{\tau,\tau+\ell} = \int_{-\infty}^{\infty} f_{N*}(\sqrt{h_\epsilon}(X_{\tau+\ell}-u)\ )\ f_N(u|0,h_b)du$$

$$= \frac{e^{-\frac{1}{2}HX^2_{\tau+\ell}}}{\sqrt{2\pi}} \cdot \frac{\sqrt{h_\epsilon}\sqrt{h_b}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(h_\epsilon+h_b)[u-A]^2} du$$

where

$$H = h_\epsilon h_b/(h_\epsilon+h_b) \text{ and } A = X_{\tau+\ell}h_\epsilon^2/(h_\epsilon+h_b) \quad .$$

Thus

$$c_{\tau,\tau+\ell} = \frac{\sqrt{H}}{\sqrt{2\pi}} \quad e^{-\frac{1}{2}HX^2_{\tau+\ell}} = f_{N*}(\sqrt{H}\ X_{\tau+\ell}) \quad .$$

3. Outline of Simulation

While it is possible to describe how $E(\tilde{\eta}_t)$ and $V(\tilde{\eta}_t)$ behave as a function of (X,L) quite easily using (10), no tractable analytical expression for the probability of insolvency given Ω within a length of time T and of the expected waiting time to insolvency given Ω appears to exist. In particular, since we wish to describe how this probability varies with changes in Ω, monte carlo simulation is a convenient and flexible method to use.

The steps to follow in simulating values of $P_\Omega(t)$ and of $E_\Omega(t)$ are in a broad outline listed below. For each value of Ω:

(1) Calculate $q_{\tau,\tau+\ell}$, p_τ, $c_{\tau,\tau+\ell}$ for $1 \leq \ell \leq L$ and $1 \leq \tau,\tau+\ell \leq 2\tau_o$ as described in 2.4 and 2.5.

(2) Use $q_{\tau,\tau+\ell}$, p_τ, $c_{\tau,\tau+\ell}$, $\underline{k}$, and r to calculate elements of $\underline{\underline{\Sigma}}$ and of $\underline{\bar{\delta}}_t$ as shown in (6).

(3) Calculate $\underline{\underline{S}}$ as described in 3.2.

(4) Carry out the simulation routine flow diagrammed in section 3.2.

(5) Estimate $P_\Omega(t)$ and $E_\Omega(t)$ and calculate the variances of these estimates.

The simulation routine consists of repeating the following steps a large number of times, say R times:

(i) Given t, generate a sequence of t standardized Normal random deviates, $\{u_\tau, \tau=1,2,\ldots,t\}$;

(ii) use $\bar{\underline{\delta}}_t$ from (6) and a matrix $\underline{\underline{S}}$ such that $\underline{\underline{S}}\,\underline{\underline{S}}^t = \underline{\underline{\Sigma}}$ to transform $\underline{u}_t \equiv (u_1, u_2 \ldots u_t)^t$ into a simulated observation of $\underline{\delta}_t$. (We show how to compute $\underline{\underline{S}}$ in 3.2).

(iii) check to see if there is a $\delta_\tau \leq 0$, $1 \leq \tau \leq t$. If so, record that ruin occurred on this particular replication. Record also the <u>smallest</u> τ for which $\delta_\tau \leq 0$ if ruin did occur*

(iv) Repeat (i), (ii), and (iii) R times.

To estimate $P_\Omega(t)$ from the results of the simulation routine we let

$$x_i = \begin{cases} 1 & \text{if insolvency occurs on the ith replication} \\ 0 & \text{otherwise} \end{cases}$$

and regard $\{x_i, i=1,2,\ldots,R\}$ as a sequence of (independent) observations generated by a Bernoulli process with parameter $P_\Omega(t)$. An unbiased estimate of $P_\Omega(t)$ is

$$\bar{x} = \frac{1}{R} \sum_{i=1}^{R} x_i, \tag{25}$$

and an unbiased estimate of the variance $V(\bar{x})$ of $\tilde{\bar{x}}$ is

$$\tilde{V}(\bar{x}) = \frac{1}{R-1} \sum_{i=1}^{R} (x_i - \bar{x})^2$$

---

*We will use a sequence of such τ's to estimate the expected waiting time $E_\Omega(t)$ to insolvency conditional on ruin occurring for some $1 \leq \tau \leq t$

We estimate $E_{\Omega}(t)$ and the variance of this estimate in a similar fashion: let

$$w_i = \begin{cases} \tau_i & \text{if} \quad \text{insolvency occurs at } 1 \leq \tau_i \leq t \\ 0 & \quad\quad \text{otherwise} \end{cases}$$

and regard $\{\tilde{w}_i, i=1,2,\ldots,R\}$ as a sequence of independent, identically distributed random variables. Then an unbiased estimate of $E_{\Omega}(t)$ is

$$\bar{w} = \frac{1}{R} \sum_{i=1}^{R} w_i \tag{26}$$

and an unbiased estimate of the variance $V(\hat{\bar{w}})$ of $\tilde{\bar{w}}$ is

$$V(\tilde{\bar{w}}) = \frac{1}{R-1} \sum_{i=1}^{R} (w_i - \bar{w})^2 . \tag{27}$$

3.2 Calculation of $\underline{\underline{S}}$

In order to carry out the simulation, we must find $\underline{\underline{S}}$ from $\underline{\underline{\Sigma}}$, which in turn is given in terms of β, r, k, $\underline{\underline{H}}$ and $\underline{\underline{Y}}$ in (21b). Since the order t of $\underline{\underline{\Sigma}}$ required during the course of the simulation is in general very large (e.g. $t = 500$ is not unreasonable) we will show how cyclicality of elements of $\underline{\underline{\Sigma}}$ may be exploited to make calculation of $\underline{\underline{S}}$ not only computationally feasible, but also analytically simple.

the application of

In Part II of this paper we show that the covariance terms $\sigma_{\tau\tau'}$ of $\underline{\underline{\Sigma}}$ are periodic in the sense that there is an integer τ_o such that

$$\sigma_{\tau\tau'} = \sigma_{\tau+\kappa\tau_o, \tau'+\kappa\tau_o} \tag{28}$$

for all $\kappa=0,1,2,\ldots$ such that $1 \leq \tau+\kappa\tau_o \leq t$ and $1 \leq \tau'+\kappa\tau_o \leq t$. When the order t of $\underline{\underline{\Sigma}}$ is a multiple of τ_o (28) allows us to write $\underline{\underline{\Sigma}}$ in the form

$$\underline{\underline{\Sigma}} = \begin{bmatrix} \underline{\underline{B}} & \underline{\underline{C}} & & & & & & \\ \underline{\underline{C}}^t & \underline{\underline{B}} & \underline{\underline{C}} & & & & \underline{\underline{0}} & \\ & \underline{\underline{C}}^t & \underline{\underline{B}} & \underline{\underline{C}} & & & & \\ & & \cdot & \cdot & \cdot & & & \\ & & & \cdot & \cdot & \cdot & & \\ & & & & \cdot & \cdot & \cdot & \\ & & & & & \cdot & \cdot & \cdot \\ & \underline{\underline{0}} & & & & \underline{\underline{C}}^t & \underline{\underline{B}} & \underline{\underline{C}} \\ & & & & & & \underline{\underline{C}}^t & \underline{\underline{B}} \end{bmatrix} \tag{29}$$

where

$$\underline{\underline{B}} = \begin{bmatrix} \sigma_{\delta_1} & \cdot & \cdot & \cdot & \sigma_{1\tau_o} \\ \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ \sigma_{\tau_o 1} & \cdot & \cdot & \cdot & \sigma_{\delta_{\tau_o}} \end{bmatrix} \quad \text{and} \quad \underline{\underline{C}} = \begin{bmatrix} \sigma_{1,\tau_o+1} & \cdot & \cdot & \cdot & \sigma_{1,2\tau_o} \\ \cdot & \cdot & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & \cdot & \cdot \\ \sigma_{\tau_o,\tau_o+1} & \cdot & \cdot & \cdot & \sigma_{2\tau_o,2\tau_o} \end{bmatrix} \tag{30}$$

We show below that $\underline{\underline{S}}$ of any order that is a multiple of τ_o has these properties provided (28) is true:

(i) To find $\underline{\underline{S}}$ we must diagonalize at most four positive definite symmetric matrices--two of order $2\tau_o$ and one of order τ_o. (When the order of $\underline{\underline{S}}$ is an even multiple of τ_o, only two matrices of order $2\tau_o$ and one of order τ_o need to be diagonalized.)

(ii) The matrix $\underline{\underline{S}} = (\underline{\underline{U}}\ \underline{\underline{M}})^t\ \underline{\underline{\Lambda}}^{\frac{1}{2}}$, where $\underline{\underline{U}}, \underline{\underline{M}}$ and $\underline{\underline{\Lambda}}^{\frac{1}{2}}$ are defined in terms of matrices derived in the course of the diagonalizations mentioned in (i).

Exhibit 2 outlines an easy method for calculating all but the last $2\tau_o$ rows of $\underline{\underline{S}}^t$ in terms of $\underline{\underline{B}}$, $\underline{\underline{C}}$, the orthogonal matrices $\underline{\underline{P}}$ and $\underline{\underline{Q}}$ that diagonalize $\underline{\underline{B}}$ and $\underline{\underline{C}}$ respectively, and the diagonal matrices $\underline{\underline{\Delta}}$ and $\underline{\underline{D}}$ defined in (33) and (38) below.

<u>PROOF</u>: We shall deal with the case when the order of $\underline{\underline{\Sigma}}$ is an odd multiple of τ_o. The modification necessary when the order is an even multiple of τ_o will become evident. Since the matrix

$$\underline{\underline{E}} \equiv \begin{bmatrix} \underline{\underline{B}} & \underline{\underline{C}} \\ \underline{\underline{C}}^t & \underline{\underline{B}} \end{bmatrix} \tag{31}$$

is positive definite symmetric, there is an orthogonal transformation $\underline{\underline{Q}}$ of dimension $(2\tau_o \times 2\tau_o)$ which diagonalizes $\underline{\underline{E}}$:

$$\underline{\underline{Q}}\,\underline{\underline{E}}\,\underline{\underline{Q}}^t = \begin{bmatrix} \lambda_1 & & & \underline{\underline{0}} \\ & \cdot & & \\ & & \cdot & \\ \underline{\underline{0}} & & & \lambda_{2\tau_o} \end{bmatrix} \equiv \underline{\underline{\Delta}} \tag{32}$$

Partitioning $\underline{\underline{\Delta}}$ into $(\tau_o \times \tau_o)$ matrices we may write

$$\underline{\underline{\Delta}} = \begin{bmatrix} \underline{\underline{\Delta}}_{11} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{\Delta}}_{22} \end{bmatrix} \tag{33}$$

Defining the $(t \times t)$ orthogonal matrix $\underline{\underline{M}}$ composed of $\underline{\underline{Q}}$'s plus a $(\tau_o \times \tau_o)$ identity matrix down the diagonal and zeros elsewhere,

$$\underline{\underline{M}} = \begin{bmatrix} \underline{\underline{Q}} & & & & \underline{\underline{0}} \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ \underline{\underline{0}} & & & \underline{\underline{Q}} & \\ & & & & \underline{\underline{I}} \end{bmatrix}, \tag{34}$$

it follows that

$$\underline{\underline{M}}\,\underline{\underline{\Sigma}}\,\underline{\underline{M}}^t = \begin{bmatrix} \underline{\underline{\Delta}}_{11} & & & & & \\ & \begin{bmatrix} \underline{\underline{\Delta}}_{22} & \underline{\underline{C}} \\ \underline{\underline{C}}^t & \underline{\underline{\Delta}}_{11} \end{bmatrix} & & & \underline{\underline{0}} & \\ & & \begin{bmatrix} \underline{\underline{\Delta}}_{22} & \underline{\underline{C}} \\ \underline{\underline{C}}^t & \underline{\underline{\Delta}}_{11} \end{bmatrix} & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \cdot \\ & \underline{\underline{0}} & & & & \begin{bmatrix} \underline{\underline{\Delta}}_{22} & \underline{\underline{C}} \\ \underline{\underline{C}}^t & \underline{\underline{B}} \end{bmatrix} \end{bmatrix} \tag{35}$$

Letting

$$\underline{\underline{V}} \equiv \begin{bmatrix} \underline{\underline{\Delta}}_{22} & \underline{\underline{C}} \\ \underline{\underline{C}}^t & \underline{\underline{\Delta}}_{11} \end{bmatrix} \quad \text{and} \quad \underline{\underline{V}}_o \equiv \begin{bmatrix} \underline{\underline{\Delta}}_{22} & \underline{\underline{C}} \\ \underline{\underline{C}}^t & \underline{\underline{B}} \end{bmatrix} \tag{36}$$

we may write

$$\underline{\underline{M}}\,\underline{\underline{\Sigma}}\,\underline{\underline{M}}^t = \begin{bmatrix} \underline{\underline{\Delta}}_{11} & & & & \\ & \underline{\underline{V}} & & \underline{\underline{C}} & \\ & & \underline{\underline{V}} & & \\ & \underline{\underline{0}} & & \ddots & \\ & & & & \underline{\underline{V}}_o \end{bmatrix} \tag{37}$$

Since $\underline{\underline{V}}$ is also positive definite symmetric there is a $(2\tau_o \times 2\tau_o)$ orthogonal matrix $\underline{\underline{P}}$ which diagonalizes $\underline{\underline{V}}$; that is,

$$\underline{\underline{P}}\,\underline{\underline{V}}\,\underline{\underline{P}}^t = \begin{bmatrix} d_1 & & \underline{\underline{0}} \\ & \ddots & \\ \underline{\underline{0}} & & d_{2\tau_o} \end{bmatrix} \equiv \underline{\underline{D}} \; . \tag{38}$$

Similarly, let $\underline{\underline{P}}_o$ be the orthogonal matrix which diagonalizes $\underline{\underline{V}}_o$, and $\underline{\underline{D}}_o$ the corresponding diagonal matrix.

Defining

$$\underline{\underline{U}} = \begin{bmatrix} \underline{\underline{I}} & & & & \\ & \underline{\underline{P}} & & \underline{\underline{0}} & \\ & & \underline{\underline{P}} & & \\ & \underline{\underline{0}} & & \ddots & \\ & & & & \underline{\underline{P}}_o \end{bmatrix} \tag{39}$$

we may write

$$\underline{\underline{U}}\ \underline{\underline{M}}\ \underline{\underline{\Sigma}}\ (\underline{\underline{U}}\ \underline{\underline{M}})^t = \underline{\underline{\Lambda}}$$

where $\underline{\underline{\Lambda}}$ is the diagonal matrix whose diagonal elements are the characteristic roots of $\underline{\underline{\Sigma}}$.

Thus

$$\underline{\underline{\Sigma}} = [(\underline{\underline{U}}\ \underline{\underline{M}})^t\ \underline{\underline{\Lambda}}^{\frac{1}{2}}]\ [(\underline{\underline{U}}\ \underline{\underline{M}})^t\ \underline{\underline{\Lambda}}^{\frac{1}{2}}]^t \tag{40}$$

or

$$\underline{\underline{\Sigma}} = \underline{\underline{S}}\ \underline{\underline{S}}^t$$

where

$$\underline{\underline{S}} = (\underline{\underline{U}}\ \underline{\underline{M}})^t\ \underline{\underline{\Lambda}}^{\frac{1}{2}} \quad , \tag{41}$$

We may use (34) and (39) to determine $\underline{\underline{S}}$ in terms of partitioned elements of $\underline{\underline{Q}}$ and $\underline{\underline{P}}$. Partition both $\underline{\underline{P}}$ and $\underline{\underline{Q}}$ into elements composed of $(\tau_o \times \tau_o)$ matrices:

$$\underline{\underline{P}} = \begin{bmatrix} \underline{\underline{P}}_{11} & \underline{\underline{P}}_{12} \\ \underline{\underline{P}}_{21} & \underline{\underline{P}}_{22} \end{bmatrix} \qquad \underline{\underline{Q}} = \begin{bmatrix} \underline{\underline{Q}}_{11} & \underline{\underline{Q}}_{12} \\ \underline{\underline{Q}}_{21} & \underline{\underline{Q}}_{22} \end{bmatrix}$$

$$\underline{\underline{P}}_o = \begin{bmatrix} \underline{\underline{P}}_{11}^{(o)} & \underline{\underline{P}}_{12}^{(o)} \\ \underline{\underline{P}}_{21}^{(o)} & \underline{\underline{P}}_{22}^{(o)} \end{bmatrix} \tag{42}$$

Then

$$
\mathbf{U}\,\mathbf{M} = \begin{bmatrix}
\mathbf{I} & & & & & & \\
& \mathbf{P}_{11} & \mathbf{P}_{12} & & & & \\
& \mathbf{P}_{21} & \mathbf{P}_{22} & & & & \\
& & & \ddots & & & \\
& & & & & \mathbf{P}_{11}^{(o)} & \mathbf{P}_{12}^{(o)} \\
& & & & & \mathbf{P}_{21}^{(o)} & \mathbf{P}_{22}^{(o)}
\end{bmatrix}
\begin{bmatrix}
\mathbf{Q}_{11} & \mathbf{Q}_{12} & & & & \\
\mathbf{Q}_{21} & \mathbf{Q}_{22} & & & & \\
& & \ddots & & & \\
& & & \mathbf{Q}_{11} & \mathbf{Q}_{12} & \\
& & & \mathbf{Q}_{21} & \mathbf{Q}_{22} & \\
& & & & & \mathbf{I}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
\mathbf{Q}_{11} & \mathbf{Q}_{12} & & & & & & \\
(\mathbf{P}_{11}\mathbf{Q}_{21}) & (\mathbf{P}_{11}\mathbf{Q}_{22}) & (\mathbf{P}_{12}\mathbf{Q}_{11}) & (\mathbf{P}_{12}\mathbf{Q}_{12}) & & & & \\
(\mathbf{P}_{21}\mathbf{Q}_{21}) & (\mathbf{P}_{21}\mathbf{Q}_{22}) & (\mathbf{P}_{22}\mathbf{Q}_{11}) & (\mathbf{P}_{22}\mathbf{Q}_{12}) & & & & \\
& & (\mathbf{P}_{11}\mathbf{Q}_{21}) & (\mathbf{P}_{11}\mathbf{Q}_{22}) & (\mathbf{P}_{12}\mathbf{Q}_{11}) & (\mathbf{P}_{12}\mathbf{Q}_{12}) & & \\
& & (\mathbf{P}_{21}\mathbf{Q}_{21}) & (\mathbf{P}_{21}\mathbf{Q}_{22}) & (\mathbf{P}_{22}\mathbf{Q}_{11}) & (\mathbf{P}_{22}\mathbf{Q}_{12}) & & \\
& & & & \ddots & & & \\
& & & & (\mathbf{P}_{21}\mathbf{Q}_{21}) & & (\mathbf{P}_{21}\mathbf{Q}_{22}) & \mathbf{P}_{22}^{(o)}
\end{bmatrix} \tag{43}
$$

Partitioning $\mathbf{D}$ and $\mathbf{D}_o$ into elements composed of $(\tau_o \times \tau_o)$ matrices we have

$$
\mathbf{D} = \begin{bmatrix} \mathbf{D}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_{22} \end{bmatrix}
\qquad
\mathbf{D}_o = \begin{bmatrix} \mathbf{D}_{11}^{(o)} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_{22}^{(o)} \end{bmatrix} \tag{44}
$$

Remembering that

$$\underline{\underline{\Lambda}} = \begin{bmatrix} \underline{\underline{\Delta}}_{11} & & & & & \\ & \underline{\underline{D}} & & & \underline{\underline{O}} & \\ & & \cdot & & & \\ & & & \cdot & & \\ \underline{\underline{O}} & & & & \cdot & \\ & & & & & \underline{\underline{D}}_o \end{bmatrix} , \qquad (45)$$

We obtain from (41),

$$\underline{\underline{S}}^t = \begin{bmatrix} \underline{\underline{Z}} & \underline{\underline{O}} & & & & & & \\ \underline{\underline{G}} & \underline{\underline{W}} & & & & & & \\ & \underline{\underline{G}} & \underline{\underline{W}} & & & & & \\ & & \cdot & \cdot & & & & \\ & & & \cdot & \cdot & & & \\ & & & & \cdot & \cdot & & \\ & & & & & \underline{\underline{G}} & \underline{\underline{W}} & \\ & & & & & & \underline{\underline{G}}_o & \underline{\underline{W}}_o \end{bmatrix}$$

where

$$\underline{\underline{Z}} = (\underline{\underline{\Delta}}_{11}^{\frac{1}{2}}\underline{\underline{Q}}_{11} \; \underline{\underline{\Delta}}_{11}^{\frac{1}{2}}\underline{\underline{Q}}_{12}) \quad , \qquad \underline{\underline{D}}_o = \begin{bmatrix} \underline{\underline{D}}_{11_o} & \underline{\underline{O}} \\ \underline{\underline{O}} & \underline{\underline{D}}_{22_o} \end{bmatrix} ,$$

$$\underline{\underline{G}} = \begin{bmatrix} (\underline{\underline{D}}_{11}^{\frac{1}{2}}\underline{\underline{P}}_{11}\underline{\underline{Q}}_{21}) & (\underline{\underline{D}}_{11}^{\frac{1}{2}}\underline{\underline{P}}_{11}\underline{\underline{Q}}_{22}) \\ (\underline{\underline{D}}_{22}^{\frac{1}{2}}\underline{\underline{P}}_{21}\underline{\underline{Q}}_{21}) & (\underline{\underline{D}}_{22}^{\frac{1}{2}}\underline{\underline{P}}_{21}\underline{\underline{Q}}_{22}) \end{bmatrix} , \qquad \underline{\underline{G}}_o = \begin{bmatrix} (\underline{\underline{D}}_{11_o}^{\frac{1}{2}}\underline{\underline{P}}_{11}^{(o)}\underline{\underline{Q}}_{21}) & (\underline{\underline{D}}_{11_o}^{\frac{1}{2}}\underline{\underline{P}}_{11}^{(o)}\underline{\underline{Q}}_{22}) \\ (\underline{\underline{D}}_{22_o}^{\frac{1}{2}}\underline{\underline{P}}_{21}^{(o)}\underline{\underline{Q}}_{21}^{(o)}) & (\underline{\underline{D}}_{22_o}^{\frac{1}{2}}\underline{\underline{P}}_{21}^{(o)}\underline{\underline{Q}}_{22}) \end{bmatrix}$$

$$\underline{\underline{W}} = \begin{bmatrix} (\underline{\underline{D}}_{11}^{\frac{1}{2}}\underline{\underline{P}}_{12}\underline{\underline{Q}}_{11}) & (\underline{\underline{D}}_{11}^{\frac{1}{2}}\underline{\underline{P}}_{12}\underline{\underline{Q}}_{12}) \\ (\underline{\underline{D}}_{22}^{\frac{1}{2}}\underline{\underline{P}}_{22}\underline{\underline{Q}}_{11}) & (\underline{\underline{D}}_{22}^{\frac{1}{2}}\underline{\underline{P}}_{22}\underline{\underline{Q}}_{12}) \end{bmatrix} ; \qquad \underline{\underline{W}}_o = \begin{bmatrix} \underline{\underline{D}}_{11_o}^{\frac{1}{2}}\underline{\underline{P}}_{12}^{(o)} \\ \underline{\underline{D}}_{12}^{\frac{1}{2}}\underline{\underline{P}}_{22}^{(o)} \end{bmatrix}$$

This completes the proof of properties (i) and (ii) when the order of $\underline{\underline{S}}$ is an odd multiple of τ_o. When the order of $\underline{\underline{S}}$ is an even multiple of $\tau_o \underline{\underline{V}}_o = \underline{\underline{\Delta}}_{22}$ which is already a diagonal matrix. Hence the second part of (i) must be true.*

*In the course of the simulation described in Part II, we will assume that the order of $\underline{\underline{\Sigma}}$ and hence of $\underline{\underline{S}}$ is so large as to obviate the need for diagonalizing $\underline{\underline{V}}_o$ even if the order of $\underline{\underline{S}}$ is odd. Hence we have omitted description of it in Exhibit 2.

EXHIBIT 2

CALCULATION OF $\underline{\underline{S}}$

1. Find an orthogonal matrix $\underline{\underline{Q}} \equiv \begin{bmatrix} \underline{\underline{Q}}_{11} & \underline{\underline{Q}}_{12} \\ \underline{\underline{Q}}_{21} & \underline{\underline{Q}}_{22} \end{bmatrix}$ of order $2\tau_o$ that diagonalizes

$\underline{\underline{E}} \equiv \begin{bmatrix} \underline{\underline{B}} & \underline{\underline{C}} \\ \underline{\underline{C}}^t & \underline{\underline{B}} \end{bmatrix}$. That is, a $\underline{\underline{Q}}$ such that

$$\begin{bmatrix} \underline{\underline{Q}}_{11} & \underline{\underline{Q}}_{12} \\ \underline{\underline{Q}}_{21} & \underline{\underline{Q}}_{22} \end{bmatrix} \begin{bmatrix} \underline{\underline{B}} & \underline{\underline{C}} \\ \underline{\underline{C}}^t & \underline{\underline{B}} \end{bmatrix} \begin{bmatrix} \underline{\underline{Q}}^t_{11} & \underline{\underline{Q}}^t_{21} \\ \underline{\underline{Q}}^t_{12} & \underline{\underline{Q}}^t_{22} \end{bmatrix} = \begin{bmatrix} \underline{\underline{\Delta}}_{11} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{\Delta}}_{22} \end{bmatrix}$$

where $\underline{\underline{\Delta}}_{11}$ and $\underline{\underline{\Delta}}_{22}$ are diagonal matrices of order τ_o.

2. Find an orthogonal matrix $\underline{\underline{P}} \equiv \begin{bmatrix} \underline{\underline{P}}_{11} & \underline{\underline{P}}_{12} \\ \underline{\underline{P}}_{21} & \underline{\underline{P}}_{22} \end{bmatrix}$ that diagonalizes

$\underline{\underline{V}} \equiv \begin{bmatrix} \underline{\underline{\Delta}}_{22} & \underline{\underline{C}} \\ \underline{\underline{C}}^t & \underline{\underline{\Delta}}_{11} \end{bmatrix}$. This is, a $\underline{\underline{P}}$ such that

$$\begin{bmatrix} \underline{\underline{P}}_{11} & \underline{\underline{P}}_{12} \\ \underline{\underline{P}}_{21} & \underline{\underline{P}}_{22} \end{bmatrix} \begin{bmatrix} \underline{\underline{\Delta}}_{22} & \underline{\underline{C}} \\ \underline{\underline{C}}^t & \underline{\underline{\Delta}}_{11} \end{bmatrix} \begin{bmatrix} \underline{\underline{P}}^t_{11} & \underline{\underline{P}}^t_{21} \\ \underline{\underline{P}}^t_{12} & \underline{\underline{P}}^t_{22} \end{bmatrix} = \begin{bmatrix} \underline{\underline{D}}_{11} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{D}}_{22} \end{bmatrix}$$

where $\underline{\underline{D}}_{11}$ and $\underline{\underline{D}}_{22}$ are diagonal matrices of order τ_o.

(EXHIBIT 2 CONTINUED)

3. Calculate the matrices

$$\underline{\underline{Z}} = (\underline{\underline{\Delta}}_{11}^{\frac{1}{2}} \; \underline{\underline{Q}}_{11} \quad \underline{\underline{\Delta}}_{11}^{\frac{1}{2}} \; \underline{\underline{Q}}_{12})$$

$$\underline{\underline{G}} = \begin{bmatrix} (\underline{\underline{D}}_{11}^{\frac{1}{2}} \circ \underline{\underline{P}}_{11} \circ \underline{\underline{Q}}_{21}) & (\underline{\underline{D}}_{11}^{\frac{1}{2}} \circ \underline{\underline{P}}_{11} \circ \underline{\underline{Q}}_{22}) \\ (\underline{\underline{D}}_{22}^{\frac{1}{2}} \circ \underline{\underline{P}}_{21} \circ \underline{\underline{Q}}_{21}) & (\underline{\underline{D}}_{22}^{\frac{1}{2}} \circ \underline{\underline{P}}_{21} \circ \underline{\underline{Q}}_{22}) \end{bmatrix}$$

$$\underline{\underline{W}} = \begin{bmatrix} (\underline{\underline{D}}_{11}^{\frac{1}{2}} \circ \underline{\underline{P}}_{12} \circ \underline{\underline{Q}}_{11}) & (\underline{\underline{D}}_{11}^{\frac{1}{2}} \circ \underline{\underline{P}}_{12} \circ \underline{\underline{Q}}_{12}) \\ (\underline{\underline{D}}_{22}^{\frac{1}{2}} \circ \underline{\underline{P}}_{22} \circ \underline{\underline{Q}}_{11}) & (\underline{\underline{D}}_{22}^{\frac{1}{2}} \circ \underline{\underline{P}}_{22} \circ \underline{\underline{Q}}_{12}) \end{bmatrix}$$

Write all but the last $2\tau_o$ rows of $\underline{\underline{S}}^t$ as

$$\underline{\underline{S}}^t = \begin{bmatrix} \underline{\underline{Z}} & \underline{\underline{O}} & & & & \\ \underline{\underline{G}} & \underline{\underline{W}} & & & & \\ & \underline{\underline{G}} & \underline{\underline{W}} & & & \\ & & \underline{\underline{G}} & \underline{\underline{W}} & & \\ & & & \circ & \circ & \\ & & & & \circ & \circ \\ & & & & & \circ \quad \circ \end{bmatrix}$$